Filters & Impulse Response Functions

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1. Filters

Types of Filters

- Main objective: to separate the ***long-run trend*** from the ***short-run cyclical component*** of a time series $y(t)$.
- There are various approaches to achieve this:
  - Linear filter
  - Linear filter with breaks
  - Nonlinear filters
- Nonlinear filters
  - Hodrick-Prescott filter (Hodrick & Prescott, 1997)
  - Band Pass filter (Baxter & King, 1999)
  - Hamilton filter (James Hamilton, 2017)
  - ... and some others

The Hodrick-Prescott filter (HP)

- The HP filter is the most used filter in macroeconomics
- It is given by the minimization problem:

$$\min_{\tau_t} \sum_{t=1}^{T} \left\{ (y_t - \tau_t)^2 + \lambda \left[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1}) \right]^2 \right\} \tag{1}$$

- where:
  - y_t is the observed time series
  - τ_t is the smooth trend that we want to obtain
  - λ is the parameter that we set to obtain the ***desired*** smoothness in the trend

The HP filter: Special Cases

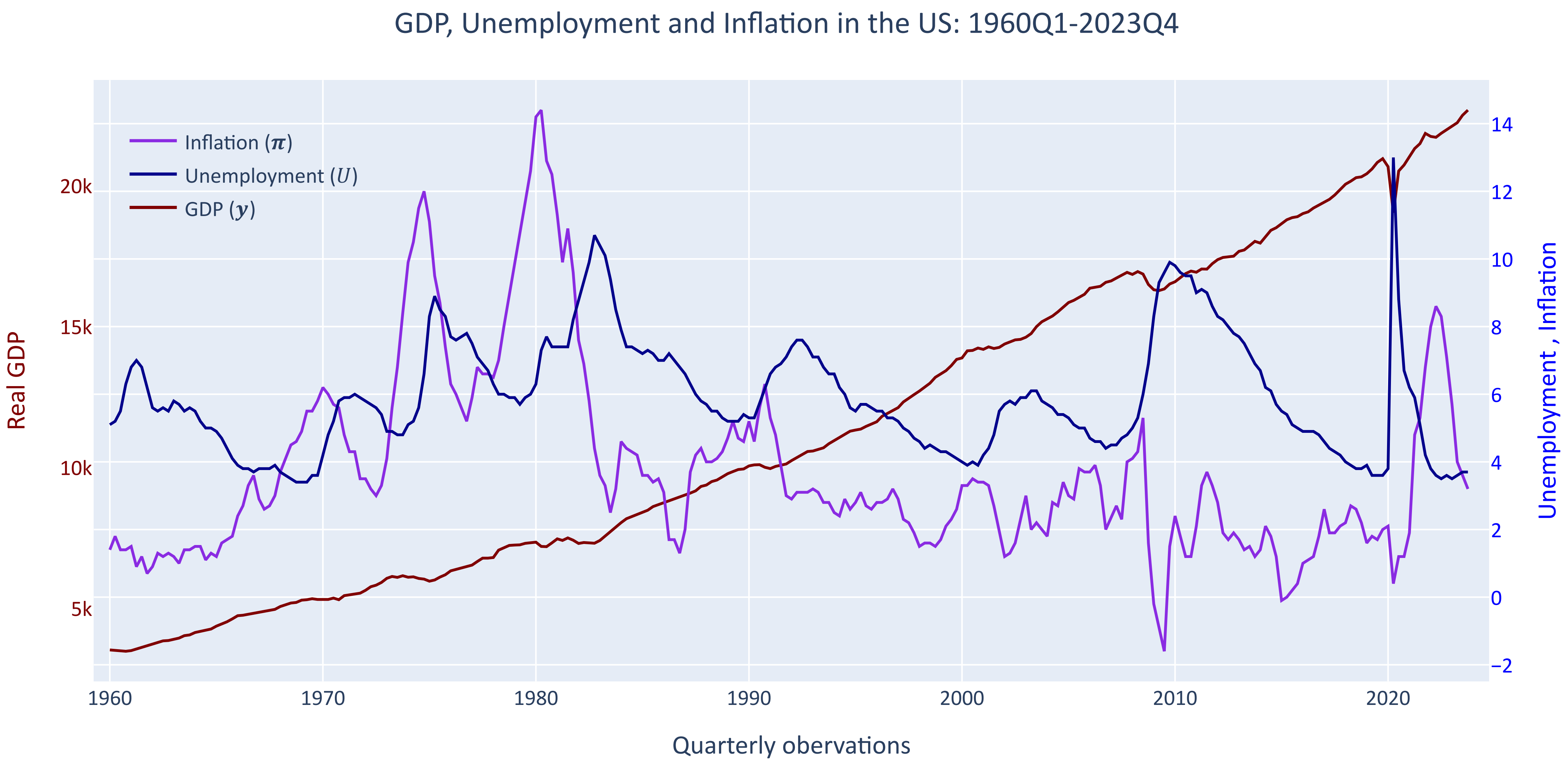
The value given to parameter λ is a choice of ours:

$$\min_{\tau_t} \sum_{t=1}^{T} \left\{ (y_t - \tau_t)^2 + \lambda [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \right\}$$

- $\lambda = 0 \Rightarrow$ trivial solution because there are ***no cycles*** : $y_t = \tau_t, \forall t$
- $\lambda \rightarrow \infty \Rightarrow$ linear trend leads to ***huge cycles*** between y_t and τ_t
- $\lambda = 1600 \Rightarrow$ duration/amplitude of cycles acceptable for ***quarterly data***
- $\lambda = 7 \Rightarrow$ duration/amplitude of cycles acceptable for ***annual data***
- There is no "unquestionable" value for λ

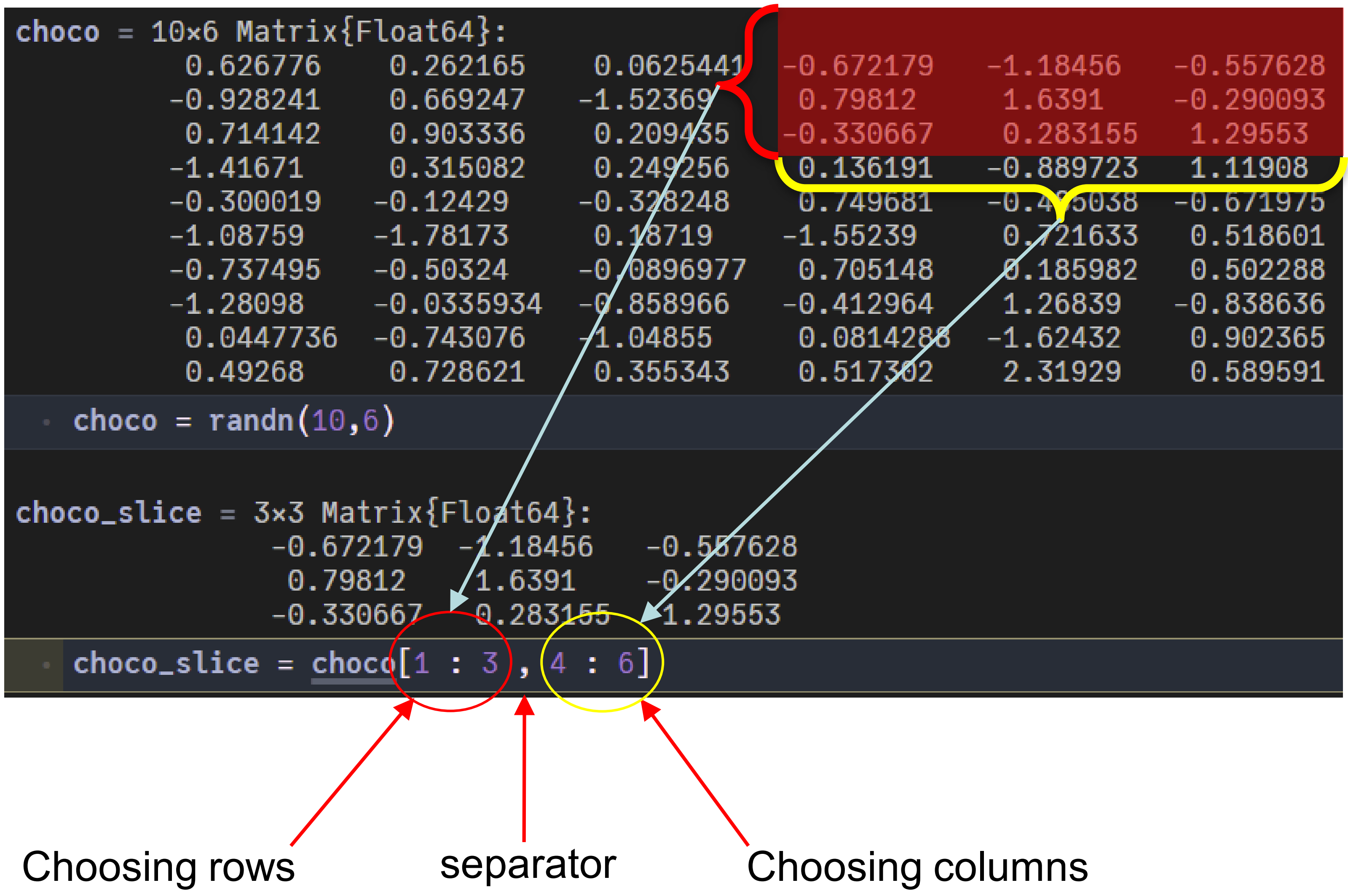
The HP Filter: an Example

- Main objective: ***obtain cycles as % deviations from the trend***
- This has an important implication:
  - Time series with a trend: ***apply logs*** to the data before extracting the trend and the cycles
  - Time series without a clear trend: ***do not apply logs*** to the data
- Quarterly data: "US_data.csv"
- A simple example:
  - Real GDP (GDP)
  - Consumer Price Index (CPI)
  - Unemployment Rate (UR)



GDP, Unemployment and Inflation in the US: 1960Q1-2023Q4
Inflation (π)
Unemployment (U)
GDP (y)
Real GDP
Unemployment , Inflation
Quarterly obervations
5k
10k
15k
20k
−2
0
2
4
6
8
10
12
14
1960
1970
1980
1990
2000
2010
2020

Dealing with rows and columns in a Matrix



Compute the HP filter: a single variable

Naming the variable with the smooth trend

Function that computes the smooth trend

The data set

Column

Smoothing parameter

```
begin
    GDP_trend = hp_filter(USdata2.lnGDP, λ)
    GDP_cycles = USdata2.lnGDP - GDP_trend
end
```

Naming the cycle

The original data

The smooth trend

Compute the HP filter: a single variable (another way)

Naming the variable with the smooth trend

Function that calculates the smooth trend

The data set used

All rows

Column 4

Smoothing parameter

```
begin
    GDP_trend = hp_filter(USdata2[:,4], λ)
    GDP_cycles = USdata2[:,4] - GDP_trend
end
```

Naming the cycles

The original data

The smooth trend

Compute the HP filter: several variables

Number of columns to iterate: $n = 1,2,3,4$

Allocating space for the trend in each variable (4 all together): a vector named as **hp_trend** with

256 rows 4 columns

All rows n columns

The **for** loop

```
begin
    hp_trend = zeros(256,4)

    for n = 1:4
        hp_trend[:,n] = hp_filter(USdata2[:,n], λ)
    end

end
```

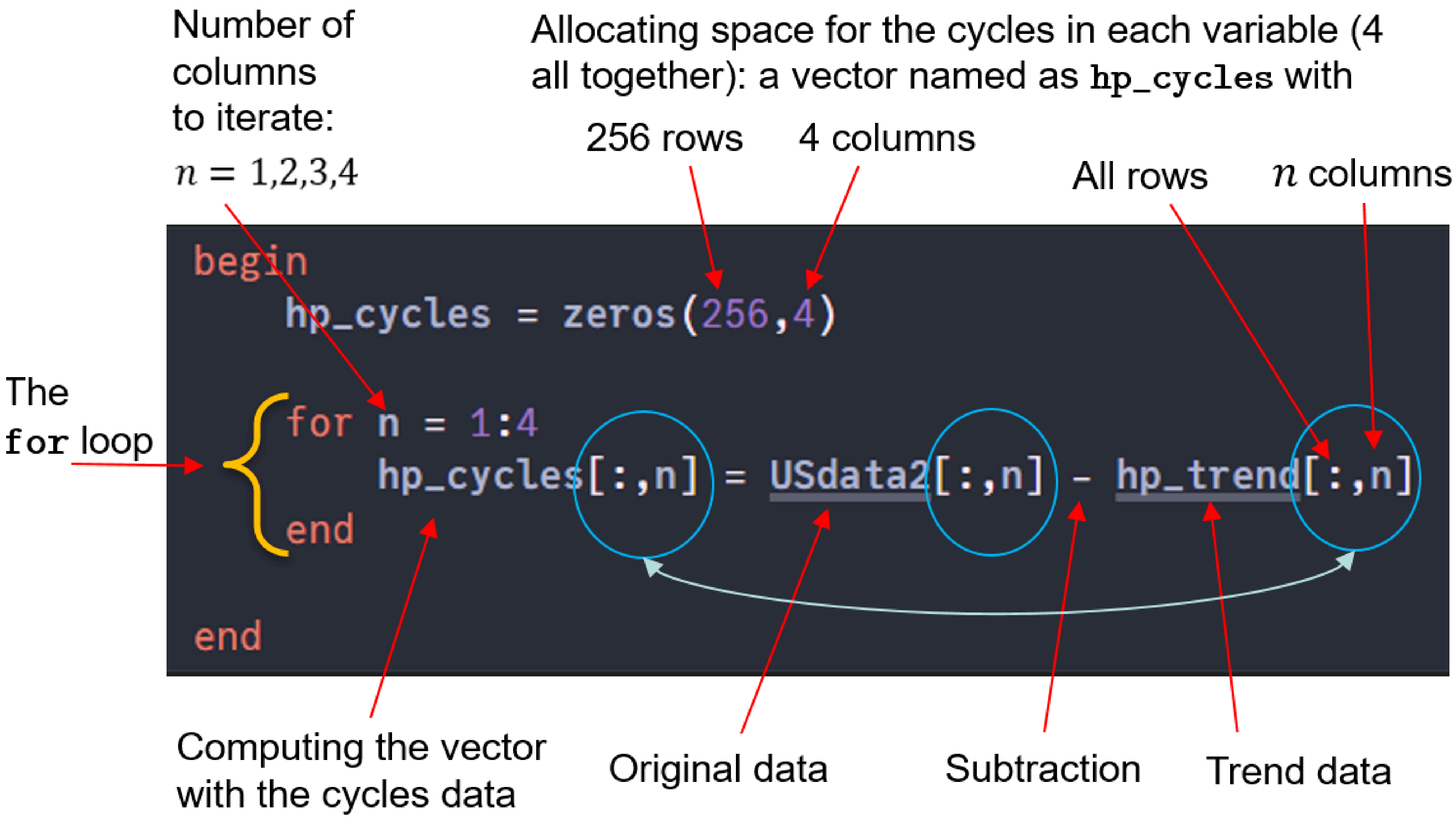
Computing the vector with the trend data

Function that calculates the smooth trend

The original data set

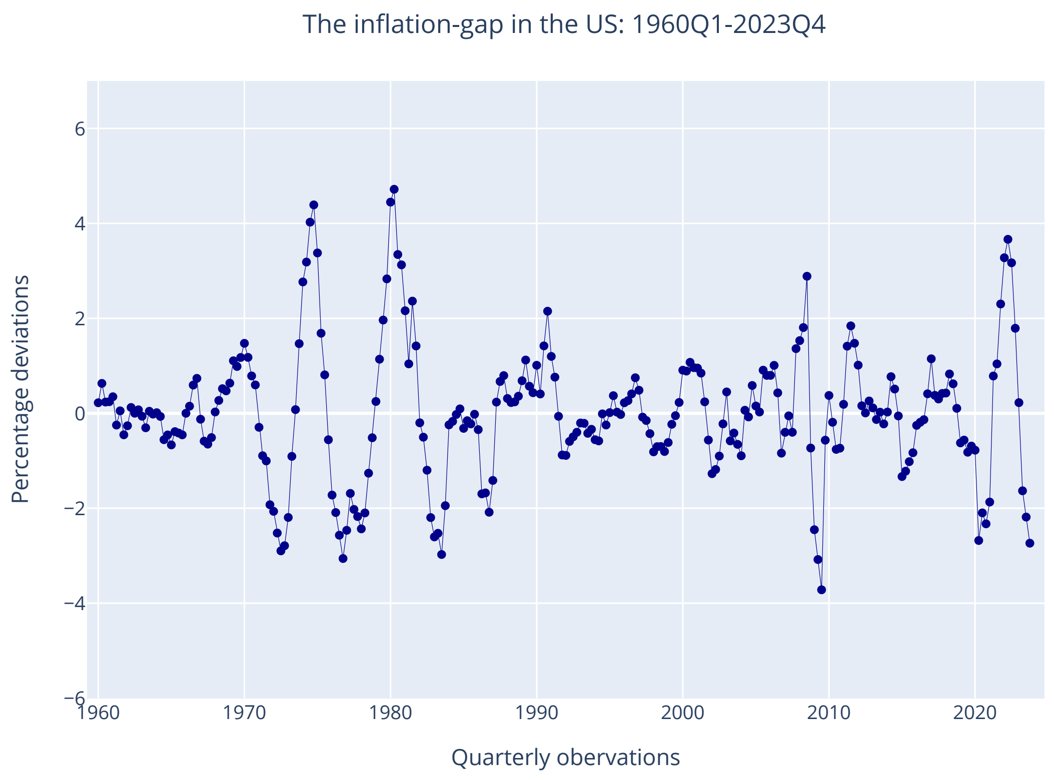
Smoothing parameter

Compute the business-cycles: several variables

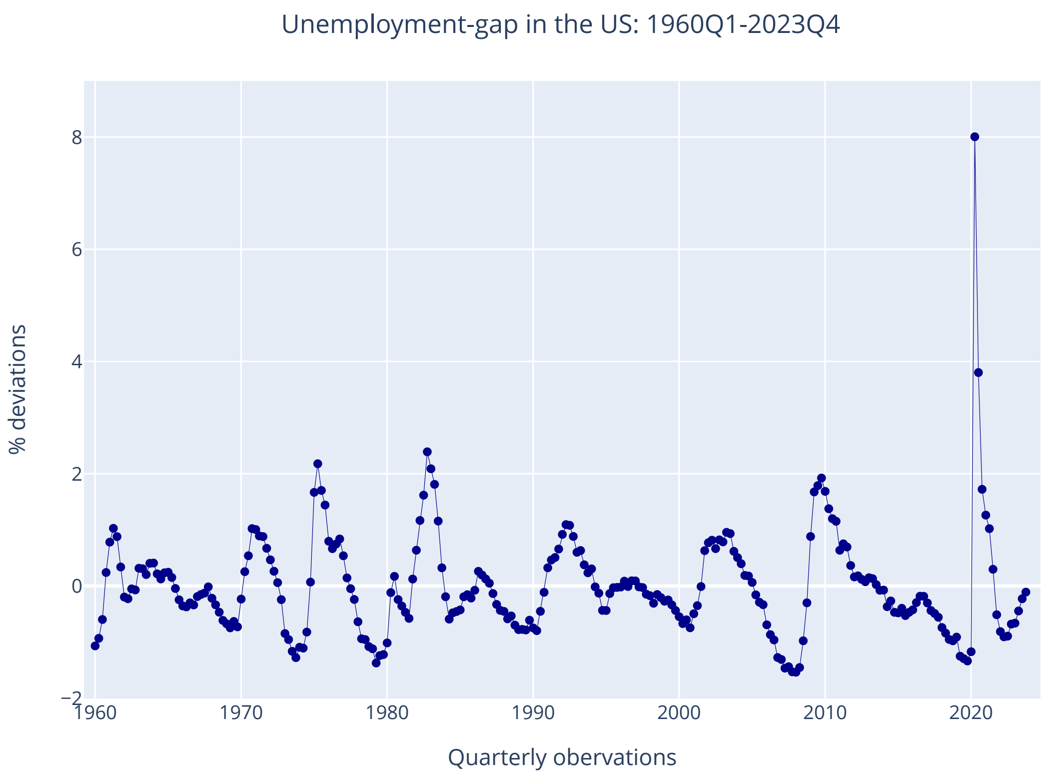


Business cycles: Inflation and Unemployment

The inflation-gap

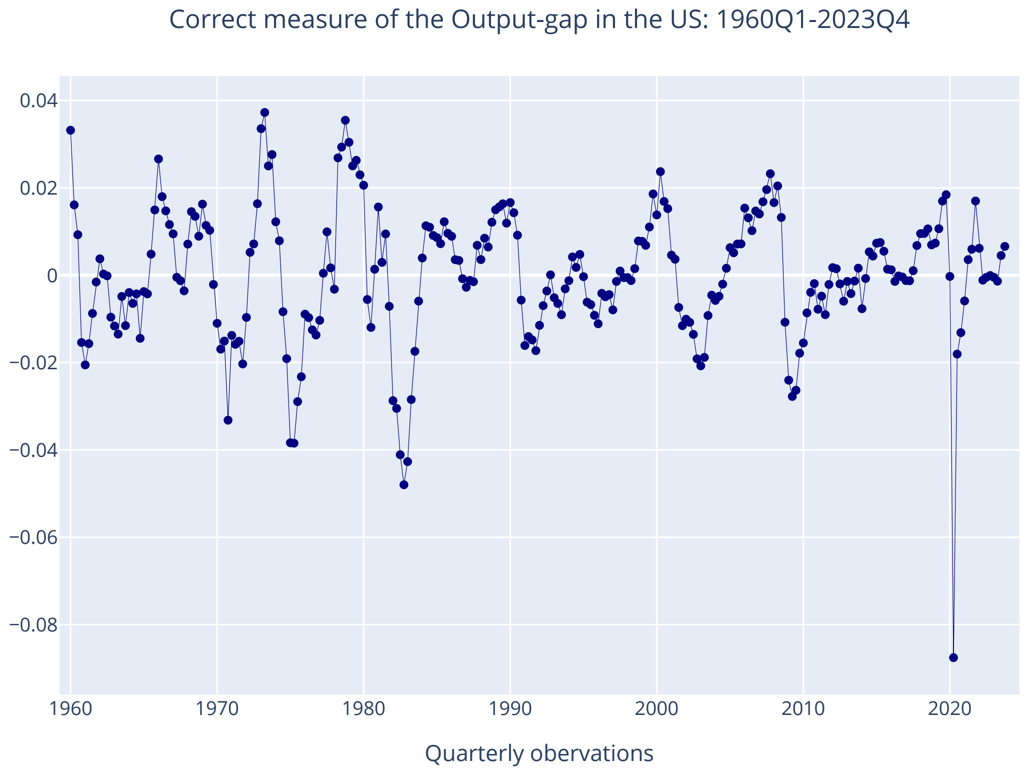


The unemployment-gap

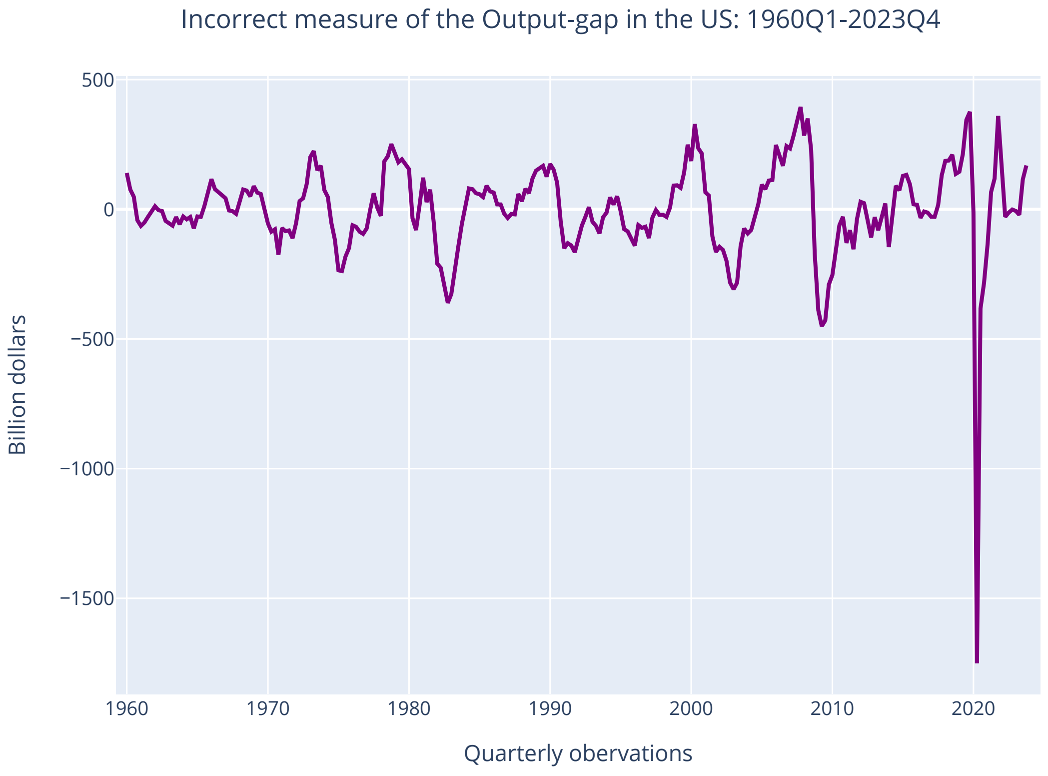


The output-gap: logs vs levels

Correctly measured: using logs



Incorrectly measured: using levels



2. Impulse Response Functions

What are IRFs?

- Impulse response functions represent the response of the endogenous variables of a given system, when one (or more than one) of its endogenous variables is hit by an exogenous shock.
- The ***nature*** of the shock can be:
    - Temporary
    - Permanent
    - Systematic
- ***Linear systems***. The magnitude of the shock does not change the stability properties of the system.
- ***Nonlinear systems***. In this case, the magnitude of the shock is of great inportance and it can change the stability of the system under consideration.

An example

- Consider the simplest case, an ***AR(1)***:

$$y_{t+1} = a y_t + \varepsilon_{t+1}, \quad \varepsilon_t \sim N(0,1) \tag{2}$$

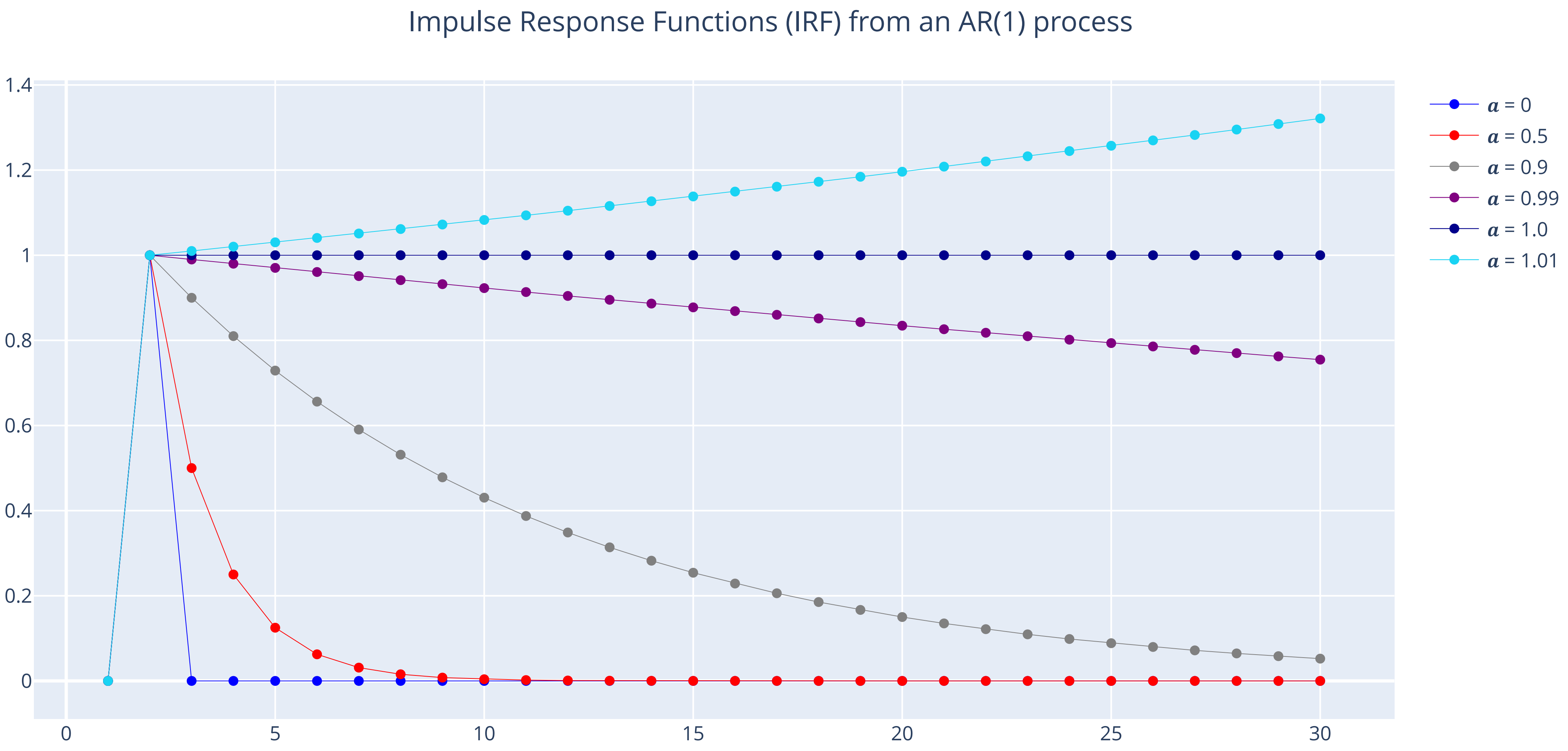
- Assume that for $t \in (1, n)$:

$$y_1 = 0\ ;\ \varepsilon_2 = 1\ ;\ \varepsilon_t = 0\ ,\ \forall t \neq 2$$

- This implies that at $t = 2 \Rightarrow y_2 = 1$. But what happens next, if there are no more shocks?
- The IRF of y provides the answer.
- The dynamics of y will depend crucially on the value of a. ***Six examples***:

$$a = \{0, 0.5, 0.9, 0.99, 1, 1.01\}$$

The IRFs of the AR(1) Process



Another example

- Consider a more sophisticated case, an ***AR(2)***:

$$y_{t+1} = a y_t + b y_{t-1} + \varepsilon_{t+1}, \quad \varepsilon_t \sim N(0,1)$$

- Assume that for $t \in (1, n)$:

$$y_1 = 0 \; ; \; y_2 = 0 \; ; \; \varepsilon_3 = 1 \; ; \; \varepsilon_t = 0 \; , \; \forall t \neq 3$$

- This implies that at

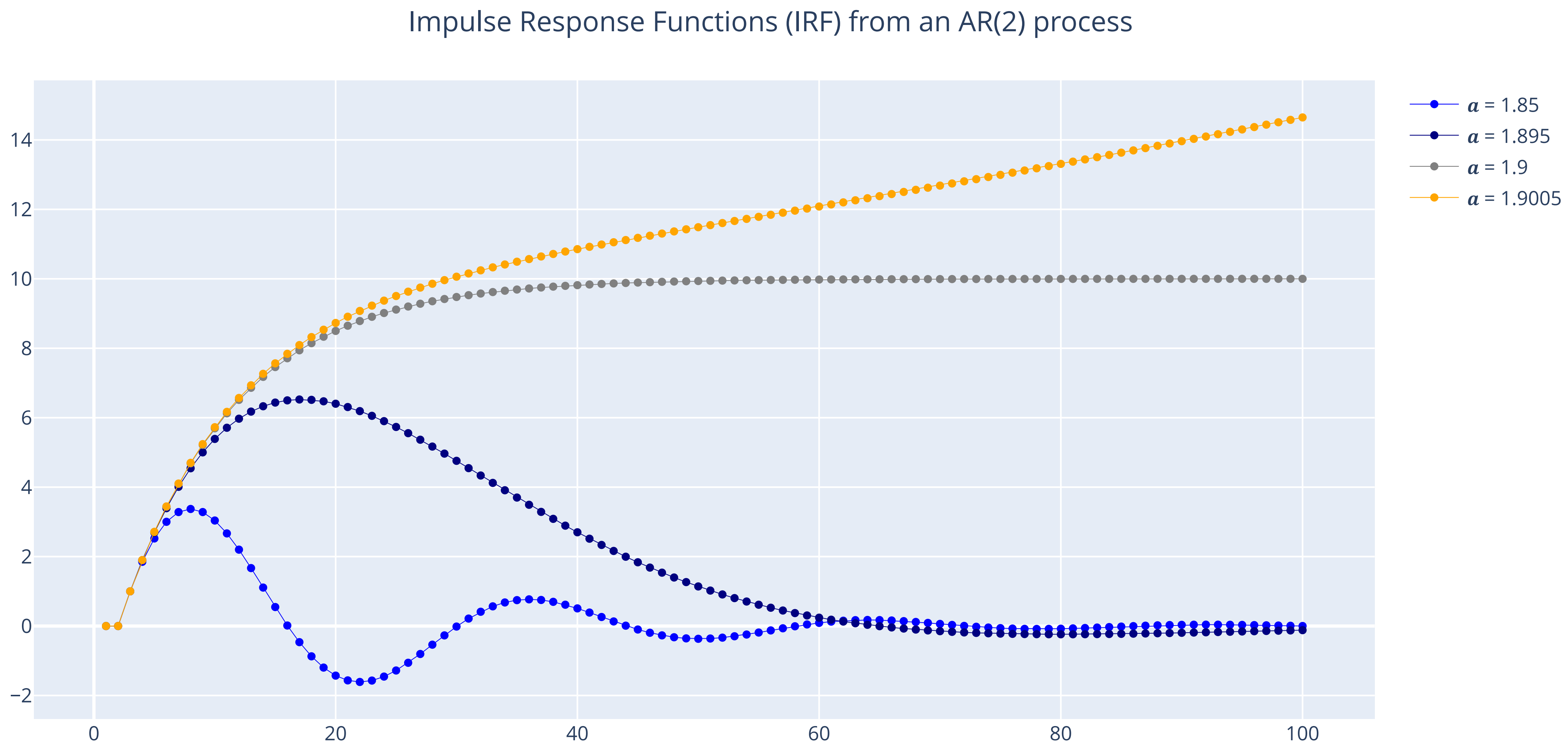
$$t = 3 \; , \; \varepsilon_3 = 1 \; \Rightarrow \; y_3 = 1.$$

- What happens next, if there are no more shocks? The IRF of y provides the answer.

- The dynamics of y will depend on the values of a and b. For simplicity consider:

$$b = -0.9 \; ; a = \{1.85, 1.895, 1.9, 1.9005\}$$

The IRFs of the AR(2) Process



More Sophisticated Examples

- A similar reasoning can be applied to our rather more general model:

$$X_{t+1} = A + BX_t + C\varepsilon_{t+1} \tag{3}$$

- .. where B, C are $n \times n$ matrices, while $X_{t+1}, X_t, A, \varepsilon_{t+1}$ are $n \times 1$ vectors.

- Consider the following ***VAR(3)*** model:

$$X_{t+1} = \begin{bmatrix} z_{t+1} \\ w_{t+1} \\ v_{t+1} \end{bmatrix}$$

- In this example we take matrices A, B and C given by:

$$A = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.97 & 0.10 & -0.05 \\ -0.3 & 0.8 & 0.05 \\ 0.01 & -0.04 & 0.96 \end{bmatrix}, \quad C = \begin{bmatrix} \color{blue}{1.0} & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}.$$

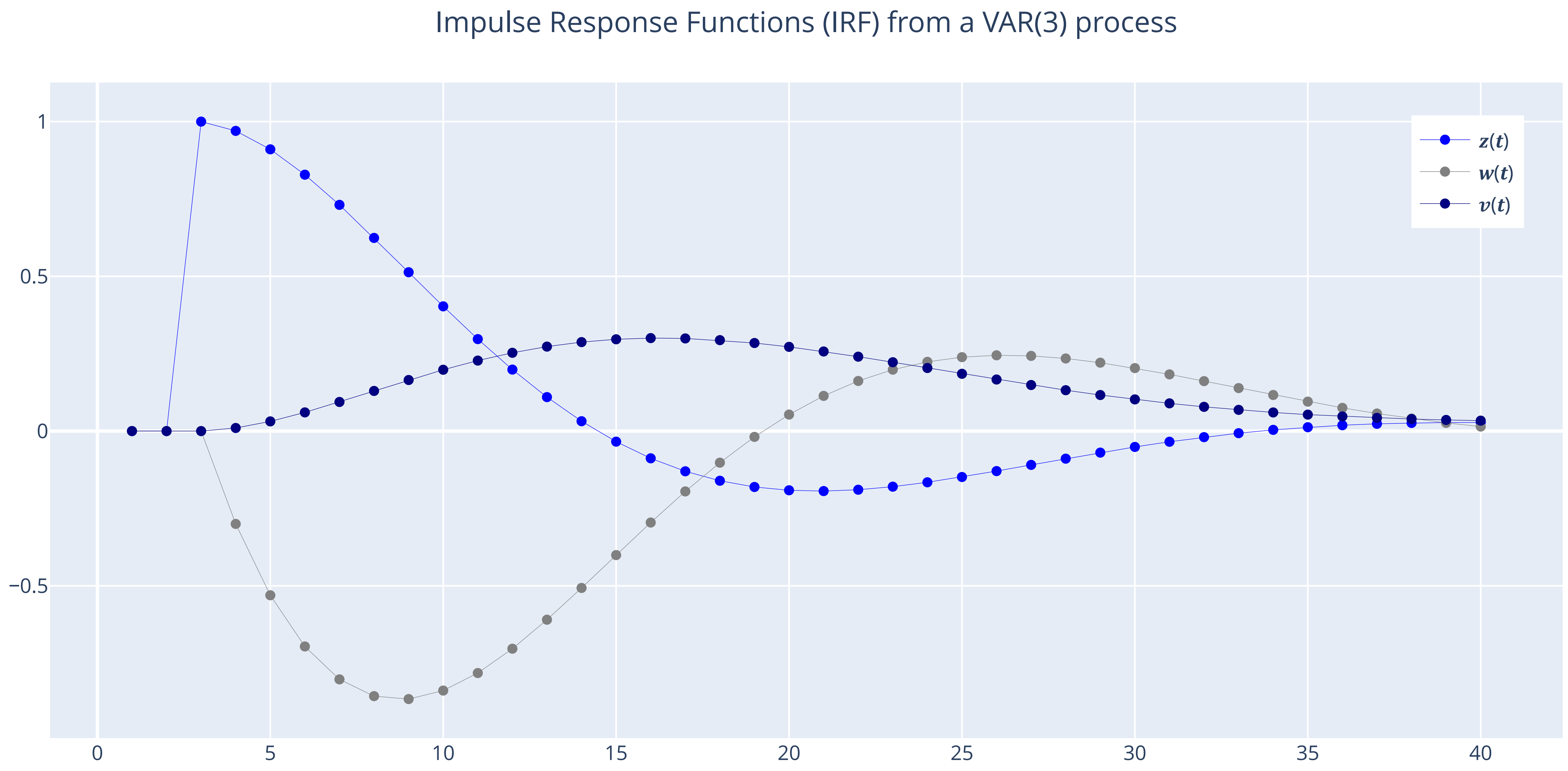
- The initial state of our system (or its initial conditions) are: $z_1 = 0, w_1 = 0$ and $v_1 = 0$, that is:

$$X_1 = [0, 0, 0]$$

- The shock only hits the variable z_t (notice the blue entry in matrix C), and we assume that the shock occurs in period $t = 3$.

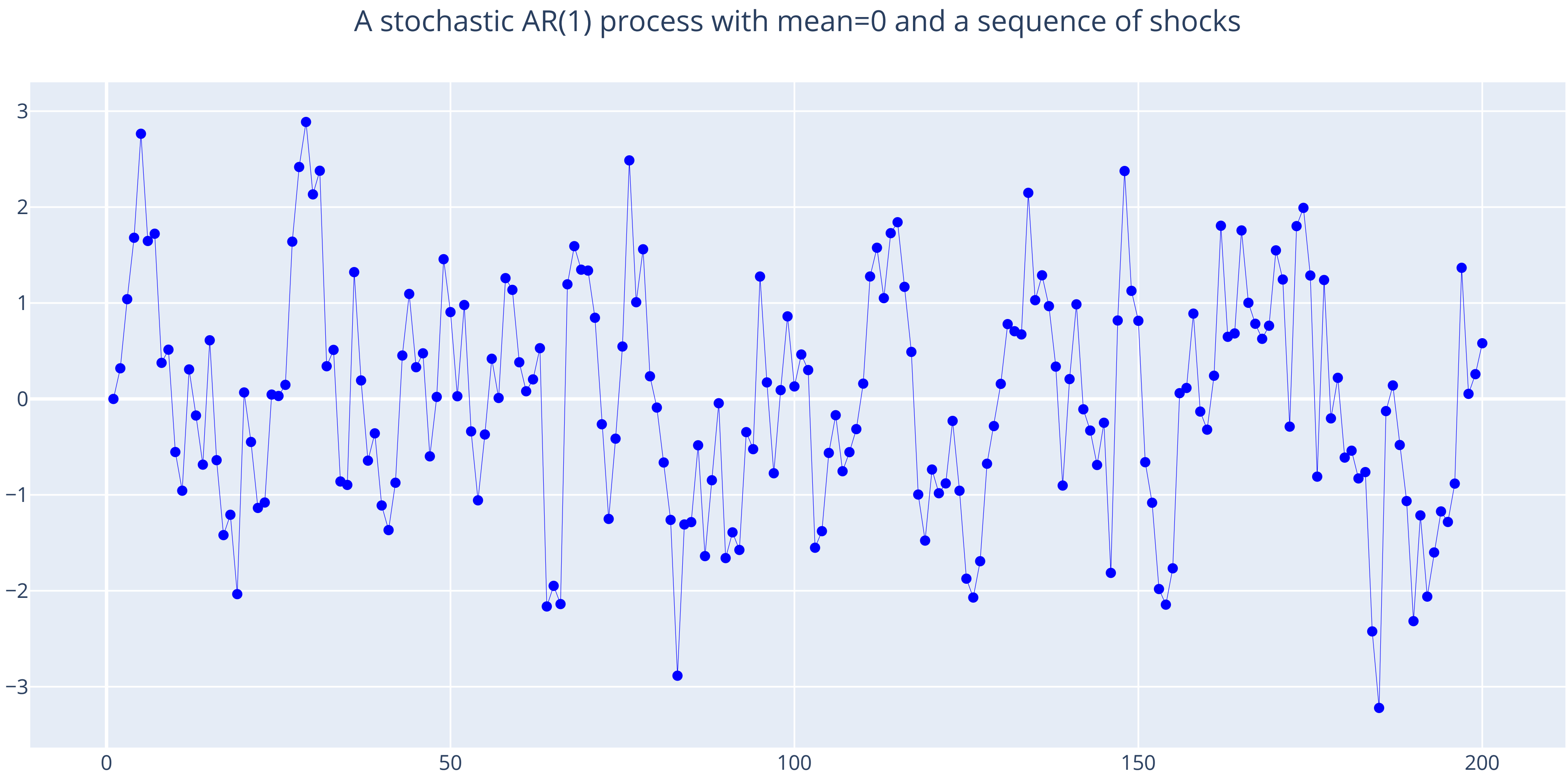
- What happens to the dynamics of the three endogenous variables? See next figure.

The IRFs of our VAR(3) Process



AR(1): A Sequence of Shocks

Consider the same AR(1) as in eq. (2). But now impose a ***sequence of 200 shocks***.



Implications of a Linear Structure

- In the previous examples, the structure of all our models ***was linear***.
- This has a crucial implication:
  The shock's magnitude did not alter the dynamics produced by the shock itself.
  - Only the structure of the model would lead to different outcomes.
- This does not usually occur if the structure of the model is ***non-linear***. In this case, the magnitude of the shock may produce different outcomes even if the system's structure remains the same.
- We do not have time to cover this particular point.
- But be careful: ***if the structure of the model is non-linear, large shocks can not be simulated … in a linearized version of the original system.***

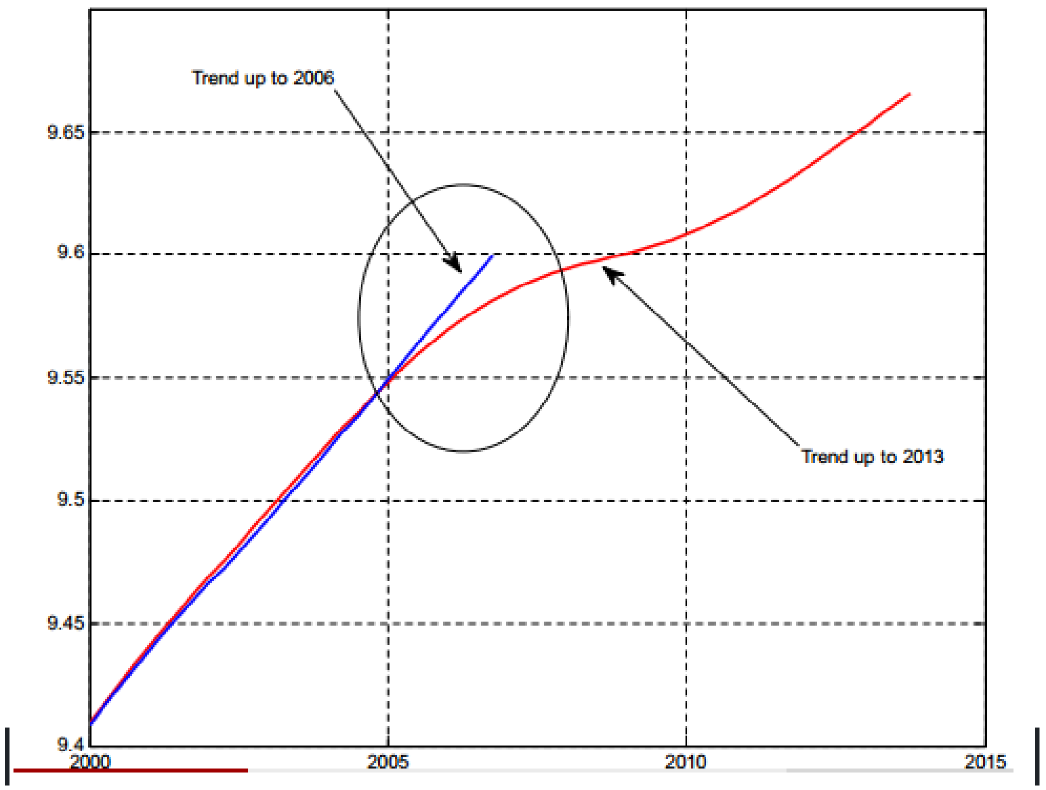
3. Important Problems

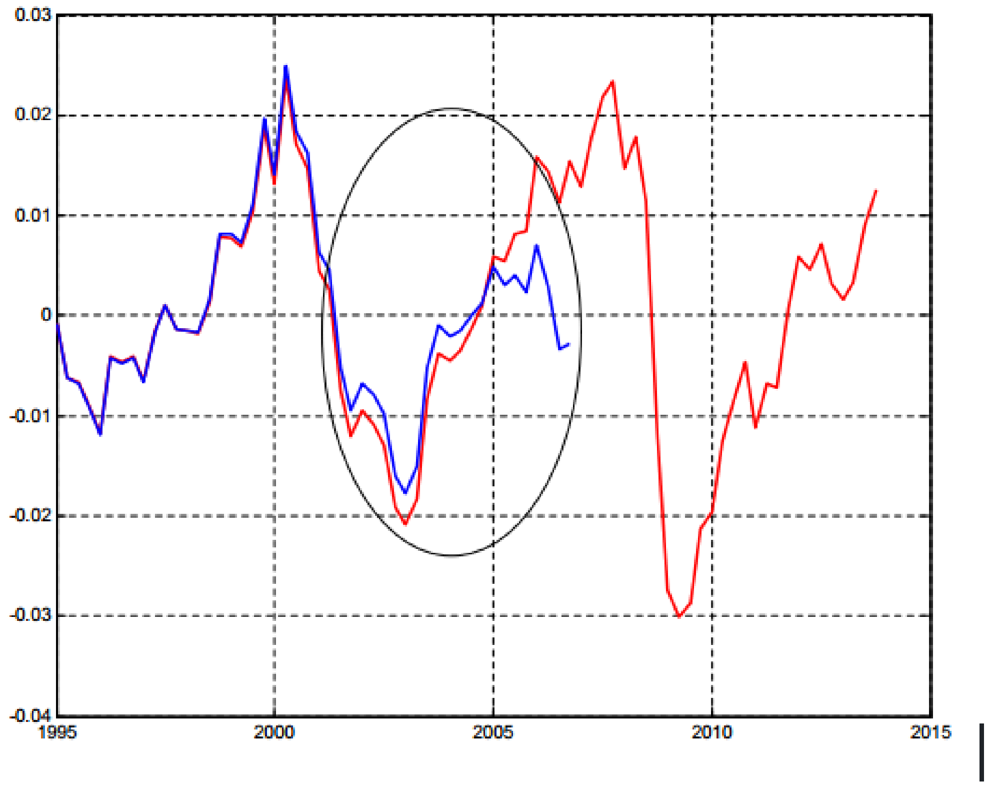
Three Major Issues

- There is ***no perfect filter*** … but the HP seems the best.
- Measuring Potential GDP (or Natural Unemployment) ***is difficult***:
  - Potential GDP is usually associated with the HP-trend in GDP … but not exclusively.
  - The Natural Rate of Unemployment is largely associated with the HP-trend in unemployment.
- All ***macroeconomic models are non-linear***:
  - Be careful with IRFs that are produced by a linearized version of the model.
  - Big shocks are difficult to be fully represented by linearization.

Limitations of the HP Filter

- ***New data*** leads to the rewriting of the history of the economy
- The HP filter is extremely useful but should be ***used with care***

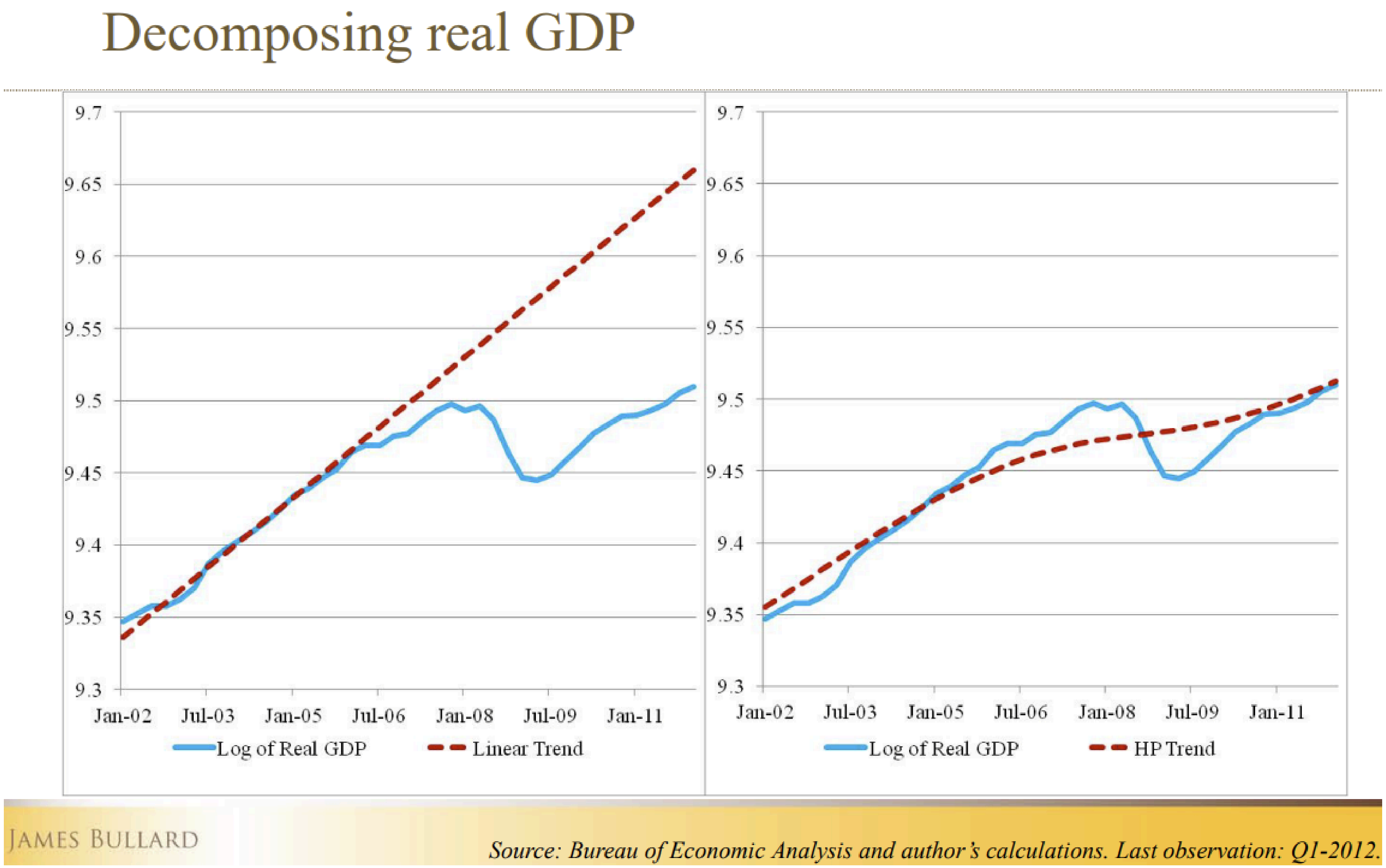




Misuses of the HP Filter

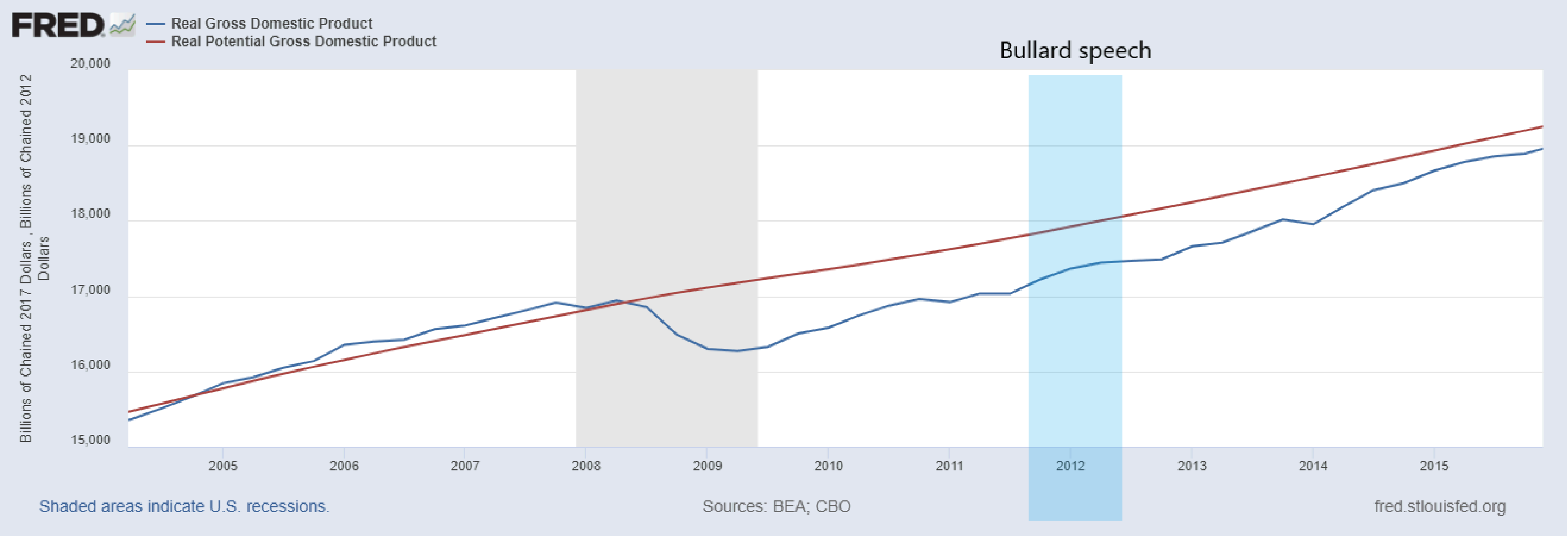
- In 2012, the US economy had an unemployment rate close to 8%, one of the highest rates since WWII.
- The Fed Funds Rate was at 0%, to stimulate the economy.
- The inflation rate was much below the target level (2%) at 0.5% and showing signs of going down.
- James Bullard (the President of the FRB of St. Louis), in a famous speech in June 2012 defended that the US economy had gone back to Potential GDP.
  - He defended that the Fed should produce a sharp increase in the Fed Funds Rate.
  - He used the HP-filter to substantiate his proposal.

The HP filter according to James Bullard



The Output-gap According to the FRB … St. Louis

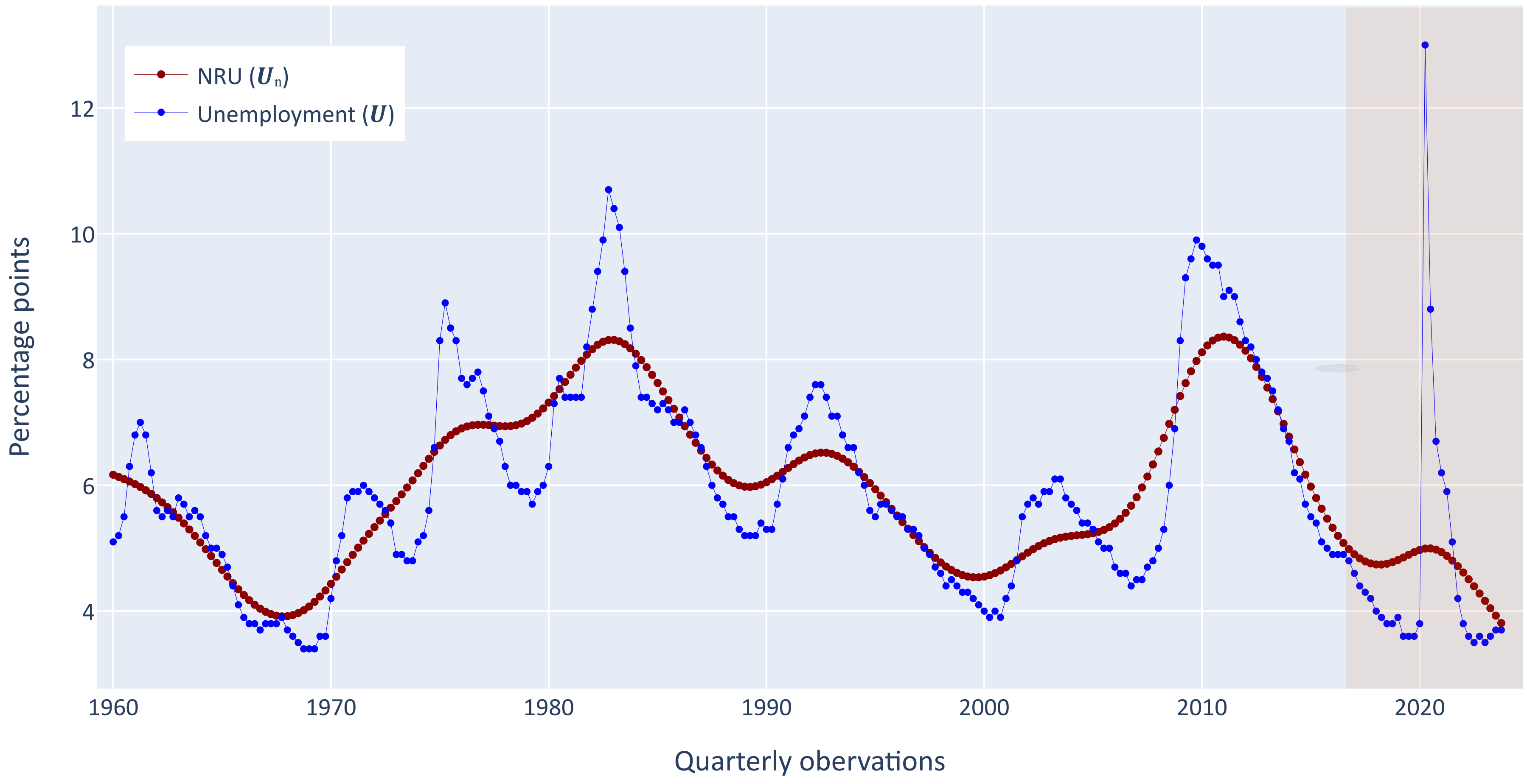
The FRB of St. Louis publishes "oficial" US data for Real GDP and Potential GDP.



The Natural Rate of Unemployment (NRU)

No, Covid-19 did not raise the NRU; no, an increase in NRU did not anticipate Covid-19!

Unemployment vs the Natural Rate of Unemployment (trend) in the US: 1960Q1-2023Q4



4. Readings

Point 1

- For this point, there is no compulsory reading.
- However, Dirk Krueger (2007). "Quantitative Macroeconomics: An Introduction" (Chapter 2), manuscript, Department of Economics University of Pennsylvania, is well suited for the material covered here.
- This text is a small one (12 pages), easy to read, and beneficial for studying the stylized facts of business cycles, mainly to understand how the Hodrick-Prescott filter is calculated. However, notice that, as mentioned, it is not compulsory reading.

Point 2

- For this point, there is no compulsory reading. However, any modern textbook on time series will cover this subject.
- At an introductory level, see sections 11.8 and 11.9 of the textbook: Diebold, F. X. (1998). *Elements of forecasting*. South-Western College Pub, Cincinnati.
- At a more advanced level, see, e.g., section 2.3.2 of the textbook: Lütkepohl, H. (2007). *New introduction to multiple time series analysis* (2nd ed.), Springer, Berlin.

Point 3

- No textbook covers the topics/controversies mentioned in this section.
- This coursework intends to provide a framework for a better understanding of these controversies at the end of the course.
- All we have to handle is:
  - A little bit of mathematics
  - A little bit of computation
  - A little bit of macroeconomics